Generalized inverses and sampling problems

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Abstract

Sampling theory is concerned with the problem of reconstructing a signal f in a Hilbert space from a given a collection of sampled values of f. If a certain decomposition of the Hilbert space is possible (in terms of the sampling and reconstruction subspaces) then a consistent reconstruction can be obtained. In this paper we treat the case in which such decomposition is not fulfilled. Under this situation, we study the *quasi-consistent reconstructions* which are an extension of the consistent reconstructions. We relate the previous concepts with generalized inverses. We also present some new results and problems regarding consistent sampling.

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1 Introduction

Sampling theory is a topic with applications in several fields such as signal and image processing, communication engineering, information theory, among others. The central idea of this theory is to recover a continuous-time function from a discrete set of samples. One of the first results in this direction was proved by Cauchy in [6]. Nevertheless, the result that had the most impact in this area is the classical Whittaker-Kotel'nikov-Shannon theorem ([17], [21], [24]) which provides conditions on a function on $\mathbb{R}$ so that it can be reconstructed from

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its sampled values at integer points. More precisely, for every $f \in L^2(\mathbb{R})$ whose Fourier transform is supported in $[-\frac{1}{2}, \frac{1}{2}]$, it holds that $f(t) = \sum_{n\in\mathbb{Z}} f(n)\text{sinc}(t-n)$ with L^2 and uniform convergences and where $\text{sinc}(t) := \frac{\sin(\pi t)}{\pi t}$.

A more general approach to sampling in an arbitrary Hilbert space is to consider the samples of the original signal f as the inner product of f with a set of sampling vectors, which span the sampling subspace $\mathcal{S}$ (see [11], [15], [22]). Hence a reconstruction of f, $\tilde{f}$, is obtained as a linear combination of a set of reconstruction vectors that span the reconstruction subspace $\mathcal{W}$. We assume that the coefficients of such reconstruction are obtained by a bounded linear transformation of the samples. This bounded linear operator will be called a filter. Observe that this framework includes the classical Whittaker-Kotel'nikov-Shannon theorem. Therefore, the sampling problem consists in selecting an appropriate filter such that the obtained reconstruction verifies some optimal criterion.

A common criterion suggested by Unser et al. in [23] is to design a reconstruction $\tilde{f}$ which is consistent with the samples, i.e., $\tilde{f}$ yields the same samples as f when it is re-injected into the system. The existence of consistent reconstructions in an arbitrary Hilbert space $\mathcal{H}$, was studied by Eldar et al. in [13]. They proved that there exists a consistent reconstruction for every $f \in \mathcal{H}$ if and only if $\mathcal{H} = \mathcal{W} + \mathcal{S}^{\perp}$. If in addition $\mathcal{W} \cap \mathcal{S}^{\perp} = \{0\}$ then the consistent reconstruction is unique. In [7], Corach et al. related the consistent sampling condition with oblique projections. In this work, we characterize the consistent filters by means of generalized inverses. Moreover, since the fact that $f, \tilde{f}$ have the same samples does not imply that they are close, we seek among the consistent reconstructions the one which is closest to f in the squared-norm sense. We shall note that this ideal reconstruction can be computed from the sequence of samples if and only if the original signal lies in a convenient subspace of $\mathcal{H}$. In this article, we also propose and study a new sampling problem. Namely, we consider the case in which two sequence of samples of the original signal are known. A natural question that arises is if it exists a simultaneous consistent reconstruction for both samples. We provide necessary and sufficient conditions that guarantee such existence. In addition, we present the general form of such recovered signals.

Another goal of this paper is to study a reconstruction-sampling scheme for the case that consistent reconstructions can not be obtained, i.e., $\mathcal{H} \neq \mathcal{W} + \mathcal{S}^{\perp}$. We define the *quasi-consistent reconstructions* as those reconstructed signals such that if they are re-injected into the system then their samples are as close as possible to the original samples. This concept is a generalization of consistent sampling. A first study of this kind of reconstructions can be found in [1]. Here, we characterize the quasi-consistent filters by means of generalized inverses. Furthermore, we obtain conditions to assure that a quasi-consistent reconstruction minimizes the squared-norm error. Moreover, if there exist infinite quasi-consistent reconstructions, then we provide two criteria for selecting a convenient one.

The paper is organized as follows: Section 2 contains a survey of results and notations used along the article. In section 3 we relate the notion of consistent reconstructions with generalized inverses. Futhermore, we determine the consistent reconstruction that minimizes the squared-norm error. In Section 4, the problem related with two samples is presented. To conclude, Section 5 is devoted to quasi-consistent reconstructions.

2 Preliminaries

In this section we present some of the results and terminology that we shall need in this paper. Throughout, Hilbert spaces are denoted by $\mathcal{H}, \mathcal{F}, \mathcal{K}, etc.$, whereas vectors in these spaces are denoted by lower-case letters. By $L(\mathcal{H}, \mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and the algebra $L(\mathcal{H}, \mathcal{H})$ is abbreviated by $L(\mathcal{H})$. For any $T \in L(\mathcal{H}, \mathcal{K})$ its range is denoted by $R(T)$, its kernel by $N(T)$ and its adjoint by T^*. In what follows, $\mathcal{S} \dot{+} \mathcal{T}$ denotes the direct sum of the closed subspaces $\mathcal{S}$ and $\mathcal{T}$. In addition, if $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{T}$ then $Q_{\mathcal{S}//\mathcal{T}}$ denotes the projection with range $\mathcal{S}$ and kernel $\mathcal{T}$. In particular, $P_{\mathcal{S}}$ indicates $Q_{\mathcal{S}//\mathcal{S}^\perp}$.

Given $A \in L(\mathcal{H}, \mathcal{K})$ with closed range the **Moore-Penrose inverse** of A, denoted by $A^\dagger$, is defined to be the unique operator X satisfying the four Penrose's equations:

1. $AXA = A$
2. $XAX = X$
3. $(AX)^* = AX$
4. $(XA)^* = XA$

Clearly, $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{N(A)^\perp}$. An operator X is called a **generalized inverse** of A, denoted by A^-, if it satisfies equation 1. In the sequel, $A[i, j, k, l]$ stands for the set of operators that verify conditions i, j, k, l. Furthermore, it holds that $A[1] = \{A^\dagger + T - A^\dagger ATAA^\dagger,\ T \in L(\mathcal{K}, \mathcal{H})\}$. For details of these matters we refer to the books [4] and [19] among many other sources.

We shall study sampling problems which are expressed as operator equations of the form $AXB = C$ with A, B, C bounded linear operators defined in convenient Hilbert spaces. In what follows the next result, that provides conditions for the solubility of this kind of equations, will play a relevant role (see [2], [20]). We shall say that the equation $AXB = C$ is **solvable** if there exists a bounded linear operator $\tilde{X}$ such that $A\tilde{X}B = C$.

Theorem 2.1 *Let $A \in L(\mathcal{H}, \mathcal{K}), B \in L(\mathcal{F}, \mathcal{G})$ and $C \in L(\mathcal{F}, \mathcal{K})$. If $R(A), R(B)$ or $R(C)$ is closed then the following conditions are equivalent:*

(1) The equation $AXB = C$ is solvable;
(2) $R(C) \subseteq R(A)$ and $R(C^*) \subseteq R(B^*)$.

Moreover, if A, B have closed ranges and $AXB = C$ is solvable then the general solution is *given by*

$$X = A^\dagger C B^\dagger + T - A^\dagger A T B B^\dagger, \tag{1}$$

for arbitrary $T \in L(\mathcal{G}, \mathcal{H})$.

The concept of frames is a useful tool to study sampling problems [3,14,25]. Recall that given $\mathcal{S}$ a closed subspace of $\mathcal{H}$, $\mathcal{F}_\mathcal{S} = \{s_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}$ is a *frame* of $\mathcal{S}$ if there exist positive constants a, b such that

$$a\|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, s_n\rangle|^2 \leq b\|f\|^2, \text{ for every } f \in \mathcal{S}.$$

Given $\mathcal{F}_\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$ a frame of $\mathcal{S}$ the operator $S \in L(l^2, \mathcal{H})$ defined by $S(\{c_n\}) = \sum_n c_n s_n \in L(l^2, \mathcal{H})$ is called the *synthesis operator* associated to $\mathcal{F}_\mathcal{S}$. Its adjoint $S^* \in L(\mathcal{H}, l^2)$ given by $S^*(f) = \{\langle f, s_n\rangle\}_{n\in\mathbb{N}}$ is called the *analysis operator* of $\mathcal{F}_\mathcal{S}$. It is well-known that both operators have closed range. For a comprehensive treatment of frame theory and its applications, the reader is referred to [8].

3 Consistent sampling

Now, we are ready to do a precise formulation of the sampling problem in abstract Hilbert spaces: let f be the original input signal which is assumed to belong to a Hilbert space $\mathcal{H}$. We consider two closed subspaces of $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{W}$, called the *sampling* and *reconstruction subspaces*, respectively. Given $\mathcal{F}_\mathcal{S} = \{s_n\}_{n\in\mathbb{N}}$ a frame of $\mathcal{S}$ with synthesis operator S, $S^*f = \{\langle f, s_n\rangle\}_{n\in\mathbb{N}}$ are the *samples* of f. We point out that we can ensure that the samples have finite energy because we are working with frames. On the other hand, we consider a frame of $\mathcal{W}$, $\mathcal{F}_\mathcal{W} = \{w_n\}_{n\in\mathbb{N}}$, with synthesis operator W. Hence, a reconstruction of f has the form

$$\tilde{f} = \sum_n c_n w_n,$$

for some coefficients $\{c_n\}_{n\in\mathbb{N}} \in l^2$ obtained from the samples S^*f under some optimality criterion. Note that the reconstruction is well-defined, i.e. the sum converges, because $\{w_n\}_{n\in\mathbb{N}}$ is a frame of $\mathcal{W}$. That is, sampling problems consist in finding a suitable $X \in L(l^2)$ (called filter) such that the reconstruction

$$\tilde{f} = WXS^*f, \tag{2}$$

has good (in some sense) approximation properties.

A well-known criterion of reconstruction is to require that the reconstructed signal be *consistent*. Consistency was proposed in [23] as follows: a reconstruction of f, $\tilde{f} \in \mathcal{W}$, is said to be a consistent reconstruction (c.r.) if and only if it yields exactly the same measurements if it is re-injected into the system. Using the formulation introduced above it is expressed as

$$S^*\tilde{f} = S^*f.$$

Clearly, the existence of a consistent reconstruction for every $f \in \mathcal{H}$ is equivalent to the solubility of the equation $S^*WXS^* = S^*$. Following the notation used in [7] we denote by

$$\begin{aligned} \mathcal{CS}(W,S) &:= \{X \in L(l^2) : WXS^*f \text{ is a c.r. for every } f \in \mathcal{H}\} \\ &= \{X \in L(l^2) : \ S^*WXS^* = S^*\}. \end{aligned} \tag{3}$$

Observe that $\mathcal{CS}(W,S)$ is not empty if and only if $\mathcal{H} = R(W) + N(S^*)$. Indeed, by Theorem 2.1, the equation $S^*WXS^* = S^*$ is solvable if and only if $R(S^*) \subseteq R(S^*W)$, i.e., if and only if $\mathcal{H} = R(W) + N(S^*)$. In the next theorem, we relate the filters in $\mathcal{CS}(W,S)$ with the generalized inverses of S^*W.

Theorem 3.1 *Let $\mathcal{H} = R(W) + N(S^*)$. Then $\mathcal{CS}(W,S) = (S^*W)[1]$.*

Proof Consider $X \in \mathcal{CS}(W,S)$. Then, $S^*WXS^* = S^*$ and so $S^*WXS^*W = S^*W$, i.e., $X \in (S^*W)[1]$. For the converse, consider $\mathcal{T}$ a closed subspace of $\mathcal{H}$ such that $\mathcal{H} = R(W) \dotplus \mathcal{T}$ and $\mathcal{T} \subseteq N(S^*)$ (for example, $\mathcal{T} := N(S^*) \cap (R(W) \cap N(S^*))^\perp$). Take $W^- \in W[1]$ such that $WW^- = Q_{R(W)//\mathcal{T}}$. Therefore, if $X \in (S^*W)[1]$ then $S^*WXS^*WW^- = S^*WW^-$ and so, since $\mathcal{T} \subseteq N(S^*)$, $S^*WXS^* = S^*$, i.e., $X \in \mathcal{CS}(W,S)$. □

Under the assumption that $\mathcal{H} = R(W) + N(S^*)$, given $f \in \mathcal{H}$ we shall denote by

$$\mathcal{C}_{W,S}(f) := \{WXS^*f : \ X \in \mathcal{CS}(W,S)\},$$

i.e., $\mathcal{C}_{W,S}(f)$ is the set of consistent reconstructions in $\mathcal{W}$ of f. Since in general the consistent reconstruction differs from the original signal, we devote the rest of this section to seek the element in $\mathcal{C}_{W,S}(f)$ which is closest to f in the squared-norm sense. As we shall see, this problem can be solved if extra hypotheses about the original signal are given. For this purpose, the next statement due to Dvorkind et al. [10] which provides a total description of $\mathcal{C}_{W,S}(f)$ will be useful.

Theorem 3.2 *Let $\mathcal{H} = R(W) + N(S^*)$. Then, $\mathcal{C}_{W,S}(f) = F^\dagger P_{R(S)} f + N(F)$, where $F = I - P_{N(S^*)}P_{R(W)}$.*

Note that $F^\dagger P_{R(S)} f = F^\dagger S(S^*S)^\dagger (S^*f)$, i.e. $F^\dagger P_{R(S)} f$ can be obtained from the samples S^*f. Moreover, since $F^\dagger P_{R(S)} f \in N(F)^\perp$, then $F^\dagger P_{R(S)} f$ is the element in $\mathcal{C}_{W,S}(f)$ with minimal norm. The filter of this minimal reconstruction was described by Corach et al. in [7], Theorem 4.2.

We consider the following minimization problem:

Proposition 3.1 *Let $\mathcal{H} = R(W) + N(S^*)$. Then*

$$\arg \min_{\tilde{f} \in \mathcal{C}_{W,S}(f)} \|f - \tilde{f}\|^2 = F^\dagger P_{R(S)} f + P_{N(F)} f. \tag{4}$$

Proof Let us start by noting that $R(F) = R(W)^\perp + R(S)$ and $N(F) = R(W) \cap N(S^*)$, i.e., $N(F) = R(F)^\perp$. By Theorem 3.2, $\min_{\tilde{f}\in\mathcal{C}_{W,S}(f)} \|f-\tilde{f}\|^2 = \min_{v\in N(F)} \|f - F^\dagger P_{R(S)} f - v\|^2$. Now, since $F^\dagger P_{R(S)} f \in R(F^\dagger) = N(F)^\perp$ and $v \in N(F)$ then

$$\min_{v\in N(F)} \|f - F^\dagger P_{R(S)} f - v\|^2 = \min_{v\in N(F)} \|P_{N(F)^\perp} f - F^\dagger P_{R(S)} f\|^2 + \|P_{N(F)} f - v\|^2,$$

and so the assertion follows. □

The main inconvenient of expression (4) is that, in general, $P_{N(F)}f$ can not be obtained from the samples S^*f by a bounded linear operator. Indeed, $P_{N(F)} = CS^*$ for some operator $C \in L(l^2, \mathcal{H})$ if and only if $N(F) = R(W)\cap N(S^*) \subseteq R(S)$ (by Theorem 2.1), i.e., $R(W)\cap N(S^*) = \{0\}$ or, equivalently if the consistent reconstruction is unique. However, as we shall see in the next proposition, if we consider the case in which f is known to lie in an appropriate subspace then the optimal reconstruction (4) can be computed from the samples. The following result can also be found in [7], Theorem 5.1. The proof presented here differs from that of Corach et al. since we do not use the notion of oblique projections.

Proposition 3.2 *Let $\mathcal{T}$ be a closed subspace of $\mathcal{H}$. For every $f \in \mathcal{T}$ the consistent reconstruction (4) can be obtained from the samples S^*f if and only if $P_\mathcal{T}(R(W)\cap N(S^*)) \subseteq P_\mathcal{T}(R(S))$.*

Proof Observe that $P_\mathcal{T}(R(W) \cap N(S^*)) \subseteq P_\mathcal{T}(R(S))$ if and only if, by Theorem 2.1, there exists $Z \in L(l^2, \mathcal{H})$ such that $P_{N(F)}P_\mathcal{T} = ZS^*P_\mathcal{T}$. Hence, for every $f \in \mathcal{T}$ we have $P_{N(F)}f = ZS^*f$ and so (4) can be obtained from the samples of f. □

4 Consistent reconstructions for two samples

This section is devoted to study the situation in which two sequence of samples of the original signal are known. We focus our attention on determine conditions for the existence of simultaneous consistent reconstructions for both samples. In addition, we provide the expression of such recovered signals. We point out that we consider the same reconstruction subspace for both sampling procedures. More precisely, we consider a reconstruction subspace $\mathcal{W}$ with synthesis operator W and two sampling subspaces $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{H}$ with synthesis operators $S, S' \in L(l^2, \mathcal{H})$, respectively.

First, we are interested in finding necessary and sufficient conditions for $\mathcal{C}_{W,S}(f) = \mathcal{C}_{W,S'}(f)$ for every $f \in \mathcal{H}$.

Proposition 4.1 *If $\mathcal{C}_{W,S}(f)$ and $\mathcal{C}_{W,S'}(f)$ are not empty then $\mathcal{C}_{W,S}(f) = \mathcal{C}_{W,S'}(f)$ for every $f \in \mathcal{H}$ if and only if $N(S^*) = N(S'^*)$.*

Proof Suppose $\mathcal{C}_{W,S}(f) = \mathcal{C}_{W,S'}(f)$ for all $f \in \mathcal{H}$ and let $f \in N(S^*)$. Then, $\tilde{f} = 0 \in \mathcal{C}_{W,S}(f)$. Thus, $\tilde{f} = 0 \in \mathcal{C}_{W,S'}(f)$ and so $S'^*f = S'^*\tilde{f} = 0$, i.e., $f \in N(S'^*)$. Conversely, suppose that $N(S^*) = N(S'^*)$ and let $\tilde{f} \in \mathcal{C}_{W,S}(f)$. Then, $S^*f = S^*\tilde{f}$, i.e., $\tilde{f} - f \in N(S^*) = N(S'^*)$. Hence, $S'^*\tilde{f} = S'^*f$ and so $\tilde{f} \in \mathcal{C}_{W,S'}(f)$. □

The following theorem provides different criterions for $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ to be not empty. Moreover, we present a fully description of this set.

Theorem 4.1 *The following conditions are equivalent:*

(1) $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ *is not empty for every* $f \in \mathcal{H}$;
(2) $\mathcal{H} = R(W) + N(S^*) \cap N(S'^*)$;
(3) for every $f \in \mathcal{H}$, $W^\dagger(\tilde{f}_S - \tilde{f}_{S'}) \in N(S^*W) + N(S'^*W)$ *where* $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$ *and* $\tilde{f}_{S'} \in \mathcal{C}_{W,S'}(f)$.

Moreover, if one of the previous conditions hold then

$$\tilde{f}_{S,S'} = \tilde{f}_S + WP_{N(S^*W)}G^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S) \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f), \tag{5}$$

where $\tilde{f}_S \in \mathcal{C}_{W,S}(f), \tilde{f}_{S'} \in \mathcal{C}_{W,S'}(f)$ *and* $G = S'^*WP_{N(S^*W)}$. *Furthermore,*

$$\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{\tilde{f}_{S,S'} + WP_{N(S^*W)}(I - G^-G)h,\ h \in \mathcal{H}\}. \tag{6}$$

Proof $1 \Leftrightarrow 2$ Suppose that $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ is not empty for every $f \in \mathcal{H}$ and consider $f \in \mathcal{H}$. Then, there exists $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Clearly, $\tilde{f} \in R(W)$. Moreover, since $S^*f = S^*\tilde{f}$ and $S'^*f = S'^*\tilde{f}$, then $z := f - \tilde{f} \in N(S^*) \cap N(S'^*)$. Then, $f = \tilde{f} + z \in R(W) + N(S^*) \cap N(S'^*)$.

Conversely, suppose that $\mathcal{H} = R(W) + N(S^*) \cap N(S'^*)$. Note that this implies that $\mathcal{C}_{W,S}(f)$ and $\mathcal{C}_{W,S'}(f)$ are not empty for every f. Now, given $f \in \mathcal{H}$ let $f = \tilde{f} + w$ with $\tilde{f} \in R(W)$ and $w \in N(S^*) \cap N(S'^*)$. Hence, $S^*f = S^*\tilde{f}$ and $S'^*f = S'^*\tilde{f}$. Therefore, $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$.

$1 \Leftrightarrow 3$ Let $f \in \mathcal{H}$, $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$ and $\tilde{f}_{S'} \in \mathcal{C}_{W,S'}(f)$. Observe that $W^\dagger\tilde{f}_S$ is a solution of $S^*Wx = S^*f$. Now, let $\tilde{f} = W\xi \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ for some $\xi \in l^2$. Note that ξ is also a solution of $S^*Wx = S^*f$. Then, $W^\dagger\tilde{f}_S - \xi \in N(S^*W)$. Analogously, $W^\dagger\tilde{f}_{S'} - \xi \in N(S'^*W)$. So, $W^\dagger(\tilde{f}_S - \tilde{f}_{S'}) = (W^\dagger\tilde{f}_S - \xi) - (W^\dagger\tilde{f}_{S'} - \xi) \in N(S^*W) + N(S'^*W)$.

Conversely, suppose that $W^\dagger(\tilde{f}_S - \tilde{f}_{S'}) = \mu - \nu$ with $\mu \in N(S'^*W), \nu \in N(S^*W)$. Then, $W^\dagger\tilde{f}_S + \nu = W^\dagger\tilde{f}_{S'} + \mu$ and, from this, $\tilde{f}_S + W\nu = \tilde{f}_{S'} + W\mu$. Let $\tilde{f} := \tilde{f}_S + W\nu \in R(W)$. Now, since $\mu \in N(S'^*W), \nu \in N(S^*W)$, then $S^*\tilde{f} = S^*f$ and $S'^*\tilde{f} = S'^*f$, i.e., $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$.

Now, let us see that $\tilde{f}_{S,S'} = \tilde{f}_S + WP_{N(S^*W)}G^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S) \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Clearly,

$\tilde{f}_{S,S'} \in R(W)$, so it only remains to show that $S^*\tilde{f}_{S,S'} = S^*f$ and $S'^*\tilde{f}_{S,S'} = S'^*f$. Now,

$$S^*\tilde{f}_{S,S'} = S^*\tilde{f}_S + S^*WP_{N(S^*W)}G^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S) = S^*\tilde{f}_S = S^*f,$$

because $\tilde{f}_S \in \mathcal{C}_{W,S}(f)$. On the other side, $S'^*\tilde{f}_{S,S'} = S'^*\tilde{f}_S + GG^\dagger S'^*(\tilde{f}_{S'} - \tilde{f}_S)$. Thus, if we show that $S'^*(\tilde{f}_{S'} - \tilde{f}_S) \in R(G)$ then the result is obtained. Now, by item 3, there exist $\nu \in N(S^*W), \mu \in N(S'^*W)$ such that $W^\dagger(\tilde{f}_{S'} - \tilde{f}_S) = \nu + \mu$. Then, $S'^*(\tilde{f}_{S'} - \tilde{f}_S) = S'^*W(W^\dagger(\tilde{f}_{S'} - \tilde{f}_S)) = S'^*W\nu \in R(G)$.

Finally, let us prove that equality (6) holds. First, note that $\tilde{f} := \tilde{f}_{S,S'} + WP_{N(S^*W)}(I - G^-G)h \in R(W)$, $S^*\tilde{f} = S^*\tilde{f}_{S,S'} = S^*f$ and $S'^*\tilde{f} = S'^*\tilde{f}_{S,S'} + S'^*WP_{N(S^*W)}(I - G^-G)h = S'^*\tilde{f}_{S,S'} + G(I - G^-G)h = S'^*\tilde{f}_{S,S'} = S'^*f$. Hence, $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$.

On the other hand, let $\tilde{f} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Then, $\tilde{f} - \tilde{f}_{S,S'} = Wh$ for some $h \in \mathcal{H}$. Moreover, since $h \in N(S^*W) \cap N(S'^*W)$ then it follows that $h \in N(G)$. So, $h = P_{N(S^*W)}(I - G^-G)h$ and the result follows. □

Remark 4.2 *We highlight that a characterization of the set $\mathcal{CS}(W,S) \cap \mathcal{CS}(W',S')$ is equivalent to study simultaneous solutions of a system of operator equations. We recommend [18] for a treatment on this topic for matrix equations. Moreover, we suggest [7] for a relationship between $\mathcal{CS}(W,S)$ and $\mathcal{CS}(W',S')$ under some range hypotheses.*

Proposition 4.3 *The set $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ has a unique reconstruction for every $f \in \mathcal{H}$ if and only if $\mathcal{H} = R(W) \dotplus N(S^*) \cap N(S'^*)$.*

Proof Let us suppose that $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{\tilde{f}_{S,S'}\}$. Then, by (6), $WP_{N(S^*W)}(I - G^-G)h = 0$, for every $h \in \mathcal{H}$, where $G = S'^*WP_{N(S^*W)}$. Now, let $v \in R(W) \cap N(S^*) \cap N(S'^*)$. Thus, $v = Wz$ for some $z \in N(S^*W)$. So, $v = WP_{N(S^*W)}z$. On the other hand, $0 = S'^*v = S'^*WP_{N(S^*W)}z = Gz$, i.e., $z \in N(G)$ and so $z = (I - G^-G)z$. Summarizing, $v = WP_{N(S^*W)}(I - G^-G)z = 0$.

Conversely, suppose that $\mathcal{H} = R(W) \dotplus N(S^*) \cap N(S'^*)$. Thus, since $WP_{N(S^*W)}(I - G^-G)h \in R(W) \cap N(S^*) \cap N(S'^*)$, we obtain that $WP_{N(S^*W)}(I - G^-G)h = 0$ for every $h \in l^2$. Now, consider $\tilde{f}_{S,S'}, \tilde{\tilde{f}}_{S,S'} \in \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$. Then, $\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'} \in R(W)$ and so $\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'} = Wz$ for some $z \in l^2$. Moreover, since $\tilde{f}_{S,S'}, \tilde{\tilde{f}}_{S,S'} \in \mathcal{C}_{W,S}(f)$, we have that $0 = S^*(\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'}) = S^*Wz$. Hence, $z \in N(S^*W)$ and so $z = P_{N(S^*W)}z$. Analogously, from $\tilde{f}_{S,S'}, \tilde{\tilde{f}}_{S,S'} \in \mathcal{C}_{W,S'}(f)$, we get that $z \in N(S'^*W)$, so $z \in N(G)$ and $z = (I - G^-G)z$. Finally, we obtain that $\tilde{f}_{S,S'} - \tilde{\tilde{f}}_{S,S'} = WP_{N(S^*W)}(I - G^-G)z = 0$, i.e, $\tilde{f}_{S,S'} = \tilde{\tilde{f}}_{S,S'}$. □

Motivated by Theorem 1 in [10], we obtain two new descriptions of the consistent reconstructions for both samples. For simplicity of notation, we denote by $\mathcal{N} = N(S'^*) \cap N(S^*)$.

Proposition 4.4 *Let $\mathcal{H} = R(W) + \mathcal{N}$. Then,*

(1) $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{Q_{L//\mathcal{N}} f : \ L \subseteq R(W)\}$.
(2) $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f) = \{J^{\dagger} P_{\mathcal{N}^{\perp}} f + \nu : \ \nu \in R(W) \cap \mathcal{N}\}$, *where* $J = I - P_{\mathcal{N}} P_{R(W)}$.

Furthermore, $J^{\dagger} P_{\mathcal{N}^{\perp}} f$ *is the reconstruction in* $\mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ *with minimal norm.*

Proof Observe that, since $\mathcal{H} = R(W) + \mathcal{N}$, then $\mathcal{C}_{W,E}(f)$ is not empty for every $f \in \mathcal{H}$ where $E \in L(\mathcal{H}, l^2)$ is such that $N(E^*) = \mathcal{N}$. Then, its straightforward that $\mathcal{C}_{W,E}(f) = \mathcal{C}_{W,S}(f) \cap \mathcal{C}_{W,S'}(f)$ for every $f \in \mathcal{H}$.

(1) The proof follows from the fact that $\tilde{f} \in \mathcal{C}_{W,E}(f)$ if and only if $\tilde{f} = Q_{L//N(E^*)} f$ with $L \subseteq R(W)$.
(2) The proof follows by Theorem 1 in [10].

Finally, since $N(J) = R(W) \cap \mathcal{N}$, it is clear that $J^{\dagger} P_{\mathcal{N}^{\perp}} f$ is the reconstruction with minimal norm. □

In the previous proposition we obtained the simultaneous consistent reconstruction with minimal norm, namely $J^{\dagger} P_{\mathcal{N}^{\perp}} f$. What is still lacking is an explicit description of this optimal recovered signal in terms of the samples.

5 Quasi-consistent reconstructions

In this section we treat the case in which $\mathcal{H} \neq R(W) + N(S^*)$, i.e., when it is not possible to find a consistent reconstruction for every $f \in \mathcal{H}$. Hence, we are interested in finding a reconstruction $\tilde{f} \in \mathcal{W}$ such that if it is re-injected into the system then the obtained measurements are as close as possible to the original samples.

Therefore, we shall say that $\tilde{f} = WXS^* f$ is a **quasi-consistent reconstruction** (q-c.r.) of f if

$$\|S^* \tilde{f} - S^* f\| \leq \|S^* \hat{f} - S^* f\|, \tag{7}$$

for every reconstruction $\hat{f} \in \mathcal{W}$ of f. In the sequel we shall denote by

$$\mathcal{QC}(W, S) := \{X \in L(l^2) : WXS^* f \text{ is q-c.r. for every } \ f \in \mathcal{H}\}.$$

Clearly, if $\mathcal{H} = R(W) + N(S^*)$ then $\mathcal{CS}(W, S) = \mathcal{QC}(W, S)$. From now on we make the assumptions: $R(W) + N(S^*)$ is a closed subspace and $S^* W \neq 0$. The first condition is equivalent to $S^* W$ has closed range (see [9], Theorem 22). We note that if $S^* W = 0$ then $\mathcal{QC}(W, S) = L(l^2)$.

The following theorem is an analogue of the characterization of $\mathcal{CS}(W, S)$ given in (3).

Theorem 5.1

$$\mathcal{QC}(W,S)=\{X\in L(l^2):\ S^*WXS^*=P_{R(S^*W)}S^*\}.$$

The resulting quasi-consistent reconstructions are

$$\tilde{f}=[W(S^*W)^\dagger+WP_{N(S^*W)}T]S^*f, \tag{8}$$

with $T\in L(l^2)$*. Moreover, there exists a unique q-c.r for every* $f\in\mathcal{H}$ *if and only if* $R(W)\cap N(S^*)=\{0\}$.

Proof Let $\hat{f}=WXS^*f$ be a reconstruction of f. Then,

$$\begin{aligned}\|S^*\hat{f}-S^*f\|^2&=\|S^*WXS^*f-S^*f\|^2\\&=\|S^*WXS^*f-P_{R(S^*W)}S^*f-(S^*f-P_{R(S^*W)}S^*f)\|^2\\&=\|S^*WXS^*f-P_{R(S^*W)}S^*f-P_{R(S^*W)^\perp}S^*f\|^2\\&=\|S^*WXS^*f-P_{R(S^*W)}S^*f\|^2+\|P_{R(S^*W)^\perp}S^*f\|^2\geq\|P_{R(S^*W)^\perp}S^*f\|^2.\end{aligned} \tag{9}$$

Now, since the equation $S^*WXS^*=P_{R(S^*W)}S^*$ is solvable then the minimum in (9) is achieved. Moreover, this minimum will be attained in those reconstructions $\tilde{f}=WXS^*f$ such that $S^*\tilde{f}=S^*WXS^*f=P_{R(S^*W)}S^*f$. Hence, in order to prove the equality (8) we shall prove that $S^*WXS^*=P_{R(S^*W)}S^*$ if and only if $WXS^*=W(S^*W)^\dagger S^*+WP_{N(S^*W)}TS^*$ for some $T\in L(l^2)$. Thus, let us suppose that $S^*WXS^*=P_{R(S^*W)}S^*$. Then, by Theorem 2.1,

$$X=(S^*W)^\dagger S^*(S^*)^\dagger+T-(S^*W)^\dagger S^*WTS^*(S^*)^\dagger, \tag{10}$$

for some $T\in L(l^2)$. Therefore, $WXS^*=W(S^*W)^\dagger S^*+WTS^*-W(S^*W)^\dagger S^*WTS^*=W(S^*W)^\dagger S^*+W(I-(S^*W)^\dagger S^*W)TS^*=W(S^*W)^\dagger S^*+WP_{N(S^*W)}TS^*$. The converse is trivial.

The unicity of the q-c.r follows from the fact that $\tilde{f}$ is a quasi-consistent reconstruction of f if and only if $S^*\tilde{f}=P_{R(S^*W)}S^*f$. □

The fact that a q-c.r. of f, $\tilde{f}$, yields the closest samples to the original ones, does not necessarily implies that $\tilde{f}$ is close to f. In the next proposition, we study this problem for the case that there exists a unique q-c.r. The first part of the next result can also be found in [1]. We include the proof for completness.

Proposition 5.1 *Let* $R(W)\cap N(S^*)=\{0\}$*. Then, the unique q-c.r. of* f *is given by* Qf *where* $Q:=W(S^*W)^\dagger S^*$ *is a projection with* $R(Q)=R(W)$*. Moreover,* $Q=P_{R(W)}$ *if and only if* $N(S^*)\subseteq R(W)^\perp\subseteq N(P_{R(S^*W)}S^*)$.

Proof Assume that $R(W)\cap N(S^*)=\{0\}$. Thus, as consequence of Theorem 5.1, the unique q-c.r. of f is given by Qf where $Q:=W(S^*W)^\dagger S^*$. Now, it is clear that $Q^2=Q$. We claim that

$R(Q) = R(W)$. Indeed, given $Wz \in R(W)$ we get $QWz = W(S^*W)^\dagger S^*Wz = WP_{N(S^*W)^\perp}z$. Now, since $N(S^*W) = N(W)$, we get $QWz = WP_{N(W)^\perp}z = Wz$, so $R(W) = R(Q)$.

Finally, suppose that $Q = P_{R(W)}$. Then, $N(S^*) \subseteq N(Q) = N(P_{R(W)}) = R(W)^\perp$. On the other hand, let $x \in R(W)^\perp = N(Q)$. Then, $0 = Qx = W(S^*W)^\dagger S^*x$ and so $0 = S^*Qx = S^*W(S^*W)^\dagger S^*x = P_{R(S^*W)}S^*x$, i.e, $x \in N(P_{R(S^*W)}S^*)$. Conversely, in order to prove that $Q = P_{R(W)}$ we shall prove that $N(P_{R(W)}) = R(W)^\perp \subseteq N(Q)$. For this, it is sufficient to show that $N(P_{R(S^*W)}S^*) \subseteq N(Q)$. Hence, given $y \in N(P_{R(S^*W)}S^*)$ we have that $0 = P_{R(S^*W)}S^*y = S^*W(S^*W)^\dagger S^*y = S^*Qy$. Therefore, $Qy = W(S^*W)^\dagger S^*y \in R(W) \cap N(S^*) = \{0\}$ and the result is proved. □

Remark 5.2 *Note that if $N(S^*) \subseteq R(W)^\perp$ then there exists $X \in L(l^2)$ such that $P_{R(W)}f = WXS^*f$ for every $f \in \mathcal{H}$. Thus, $P_{R(W)}f$ is the reconstruction of f that minimizes the squared-error $\|f - \tilde{f}\|^2$. Now, by the preceding proposition, the additional condition $R(W)^\perp \subseteq N(P_{R(S^*W)}S^*)$, guarantees that the optimal reconstruction $P_{R(W)}f$ is also quasi-consistent.*

By means of Theorem 5.1, we establish now how the notion of quasi-consistent reconstruction is related with generalized inverses. Before that, we introduce the next technical lemma.

Lemma 5.3 *Let $A \in L(\mathcal{H}, \mathcal{K}), B \in L(\mathcal{F}, \mathcal{G})$. If $AXB = 0$ for all $X \in L(\mathcal{G}, \mathcal{H})$ then $A = 0$ or $B = 0$.*

Proof Suppose that $A \neq 0$ and $B \neq 0$. Then there exist $u \in \mathcal{F}$ and $w \in \mathcal{H}$ such that $Bu = v \neq 0$ and $Aw \neq 0$. Define $X \in L(\mathcal{G}, \mathcal{H})$ such that $Xz = \langle z, v\rangle\, w$. Therefore, $AXBu = AXv = \langle v, v\rangle\, Aw \neq 0$, which contradicts the hypothesis. Then, $A = 0$ or $B = 0$. □

Theorem 5.2 *The next inclusions hold:*

$$(S^*W)[1,3] \subseteq \mathcal{QC}(W,S) \subseteq (S^*W)[1].$$

Moreover,

(1) $\mathcal{QC}(W,S) = (S^*W)[1]$ if and only if $\mathcal{H} = R(W) + N(S^*)$.
(2) $\mathcal{QC}(W,S) = (S^*W)[1,3]$ if and only if S^* is surjective.

Proof If $X \in (S^*W)[1,3]$ then $S^*WXS^* = P_{R(S^*W)}S^*$. So, $X \in \mathcal{QC}(W,S)$. On the other hand, if $X \in \mathcal{QC}(W,S)$ then $S^*WXS^* = P_{R(S^*W)}S^*$. Thus, $S^*WXS^*W = P_{R(S^*W)}S^*W = S^*W$, i.e., $X \in (S^*W)[1]$.

(1) If $\mathcal{QC}(W,S) = (S^*W)[1]$ then $X = (S^*W)^\dagger + T - (S^*W)^\dagger S^*WTS^*W(S^*W)^\dagger \in \mathcal{QC}(W,S)$ for every $T \in L(l^2)$. Therefore, by Theorem 5.1,

$$\begin{aligned} P_{R(S^*W)}S^* &= S^*WXS^* = S^*W(S^*W)^\dagger S^* + S^*WTS^* - S^*WTS^*W(S^*W)^\dagger S^* \\ &= P_{R(S^*W)}S^* + S^*WT(I - P_{R(S^*W)})S^*. \end{aligned}$$

Hence, for every $T \in L(l^2)$, $S^*WTP_{R(S^*W)^\perp}S^* = 0$. Now, by the previous lemma, we get that $P_{R(S^*W)^\perp}S^* = 0$ or, which is equivalent, $\mathcal{H} = R(W) + N(S^*)$.

Conversely, if $\mathcal{H} = R(W) + N(S^*)$ then $\mathcal{QC}(W,S) = \mathcal{CS}(W,S)$ and the assertion follows by Theorem 3.1.

(2) Suppose $\mathcal{QC}(W,S) = (S^*W)[1,3]$. Then by (10), for every $T \in L(l^2)$

$$X = (S^*W)^\dagger S^*(S^*)^\dagger + T - (S^*W)^\dagger S^*WTS^*(S^*)^\dagger \in (S^*W)[1,3].$$

Then

$$\begin{aligned} P_{R(S^*W)} &= S^*WX = S^*W(S^*W)^\dagger S^*(S^*)^\dagger + S^*WT - S^*WTS^*(S^*)^\dagger \\ &= P_{R(S^*W)}P_{R(S^*)} - S^*WTP_{R(S^*)^\perp}. \end{aligned}$$

Therefore, for every $T \in L(l^2)$, $S^*W(T - (S^*W)^\dagger)P_{R(S^*)^\perp} = 0$. Hence, by Lemma 5.3, we obtain that $P_{R(S^*)^\perp} = 0$ and so S^* is surjective. The converse is immediate. $\square$

As we have mentioned, if $R(W) \cap N(S^*) \neq \{0\}$ then there exist infinite quasi-consistent reconstructions for every $f \in \mathcal{H}$. For this situation, we present two criteria for selecting a convenient quasi-consistent reconstruction. These criteria are motivated by the work of Eldar et al. in [12].

The first method consists in minimizing the worst error between the cuasi-consistent reconstructions. That is,

$$\min_{X \in \mathcal{QC}(W,S)} \max_{\|f'\| \leq \alpha} \|WXS^*f' - f'\|^2 = \alpha^2 \min_{X \in \mathcal{QC}(W,S)} \|WXS^* - I\|^2.$$

In the second method we seek the quasi-consistent reconstruction that minimizes the worst regret, i.e.,

$$\min_{X \in \mathcal{QC}(W,S)} \max_{\|f'\| \leq \alpha} \|WXS^*f' - P_{R(W)}f'\|^2 = \alpha^2 \min_{X \in \mathcal{QC}(W,S)} \|WXS^* - P_{R(W)}\|^2.$$

In order to solve the previous problems we shall use the following results.

Lemma 5.4 ([16], Corollary 7) *If $A, B \in B(\mathcal{H})$ such that $R(A) \perp R(B)$, then*

$$\max\{\|A\|^2, \|B\|^2\} \leq \|A + B\|^2 \leq \max\{\|A\|^2, \|B\|^2\} + \|AB^*\|.$$

If in adition, $R(A^*) \perp R(B^*)$ then $\max\{\|A\|, \|B\|\} = \|A + B\|$.

Theorem 5.3 *Let $A \in L(\mathcal{H}, \mathcal{K}), B \in L(\mathcal{F}, \mathcal{G})$ with closed ranges and $C \in L(\mathcal{F}, \mathcal{K})$. If $R((B^\dagger B - I)C^*AA^\dagger) \perp R(C^*(AA^\dagger - I))$ then, for every $X \in L(\mathcal{G}, \mathcal{H})$*

$$\|AA^\dagger CB^\dagger B - C\| \leq \|AXB - C\|,$$

with equality if $X = A^\dagger C B^\dagger + T - A^\dagger A T B B^\dagger$ *for all* $T \in L(\mathcal{G}, \mathcal{H})$.

Proof Observe that $AXB - C = (AXB - AA^\dagger C) + (AA^\dagger C - C)$. Now, since $R(AXB - AA^\dagger C) \perp R(AA^\dagger C - C)$ then by the previous lemma

$$\|AXB - C\| \geq \max\{\|AXB - AA^\dagger C\|, \|AA^\dagger C - C\|\}.$$

On the other hand, $AXB - AA^\dagger C = (AXB - AA^\dagger CB^\dagger B) + (AA^\dagger CB^\dagger B - AA^\dagger C)$. As $R((AXB - AA^\dagger CB^\dagger B)^*) \perp R((AA^\dagger CB^\dagger B - AA^\dagger C)^*)$ then

$$\|AXB - AA^\dagger C\| \geq \max\{\|AXB - AA^\dagger CB^\dagger B\|, \|AA^\dagger CB^\dagger B - AA^\dagger C\|\}.$$

Summarizing, $\|AXB - C\| \geq \max\{\|AXB - AA^\dagger CB^\dagger B\|, \|AA^\dagger CB^\dagger B - AA^\dagger C\|, \|AA^\dagger C - C\|\}$. From $R(AA^\dagger CB^\dagger B - AA^\dagger C) \perp R(AA^\dagger C - C)$ and the hypothesis we have

$$\|AA^\dagger CB^\dagger B - C\| = \max\{\|AA^\dagger CB^\dagger B - AA^\dagger C\|, \|AA^\dagger C - C\|\}.$$

Finally, $\|AXB - C\| \geq \max\{\|AXB - AA^\dagger CB^\dagger B\|, \|AA^\dagger CB^\dagger B - C\|\}$ and this concludes the proof. □

Theorem 5.4 *Consider the problems*

$$\min_{X \in \mathcal{QC}(W,S)} \|WXS^* - I\|^2, \tag{11}$$

and

$$\min_{X \in \mathcal{QC}(W,S)} \|WXS^* - P_{R(W)}\|^2. \tag{12}$$

The resulting reconstruction are

$$\tilde{f} = [P_{\mathcal{M}^\perp} W (S^* W)^\dagger + P_{\mathcal{M}} (S^*)^\dagger] S^* f, \tag{13}$$

and

$$\tilde{f} = [P_{\mathcal{M}^\perp} W (S^* W)^\dagger + P_{\mathcal{M}} P_{R(W)} (S^*)^\dagger] S^* f, \tag{14}$$

respectively, where $\mathcal{M}$ *is the closed subspace* $R(WP_{N(S^*W)})$.

Proof First, note that $\mathcal{M} = R(WP_{N(S^*W)})$ is a closed subspace since $N(W) + N(S^*W) = N(S^*W)$ is closed (see [9], Theorem 22). Now, by Theorem 5.1,

$$\begin{aligned} \min_{X \in \mathcal{QC}(W,S)} \|WXS^* - I\|^2 &= \min_{L \in L(\mathcal{H})} \|W((S^*W)^\dagger + P_{N(S^*W)} L) S^* - I\|^2 \\ &= \min_{L \in L(\mathcal{H})} \|WP_{N(S^*W)} L S^* - (I - W(S^*W)^\dagger S^*)\|^2 \end{aligned}$$

In order to apply Theorem 5.3, we note that

$$P_{\mathcal{M}^\perp}(I - W(S^*W)^\dagger S^*)P_{N(S^*)}(I - S(W^*S)^\dagger W^*)P_{\mathcal{M}} = P_{\mathcal{M}^\perp}P_{N(S^*)}P_{\mathcal{M}} = 0$$

where the last equality follows from the fact that $\mathcal{M} \subseteq N(S^*)$.

Now, applying Theorems 5.1 and 5.3, we get that for all $T \in L(l^2)$

$$X = (S^*W)^\dagger + P_{N(S^*W)}[(WP_{N(S^*W)})^\dagger[I - W(S^*W)^\dagger S^*] + T - (WP_{N(S^*W)})^\dagger WP_{N(S^*W)}TS^*](S^*)^\dagger,$$

are solutions of (11). Therefore the optimal reconstruction is

$$\begin{aligned}\tilde{f} &= WXS^*f = W\left((S^*W)^\dagger + P_{N(S^*W)}(WP_{N(S^*W)})^\dagger[I - W(S^*W)^\dagger S^*](S^*)^\dagger\right)S^*f\\ &= W(S^*W)^\dagger S^*f + WP_{N(S^*W)}(WP_{N(S^*W)})^\dagger[I - W(S^*W)^\dagger S^*](S^*)^\dagger S^*f\\ &= W(S^*W)^\dagger S^*f + P_{\mathcal{M}}[(S^*)^\dagger S^*f - W(S^*W)^\dagger S^*f]\\ &= [P_{\mathcal{M}^\perp}W(S^*W)^\dagger + P_{\mathcal{M}}(S^*)^\dagger]S^*f.\end{aligned}$$

Finally, problem (12) can be solved in a similar manner. □

Remarks 5.5 *(1) The problems of minimizing the worst error and worst regret among all possible reconstructions were studied by Eldar et al. in [12].*

(2) If $(S^*W)^\dagger = W^\dagger(S^*)^\dagger$ then $W(S^*W)^\dagger S^* = WW^\dagger(S^*)^\dagger S^* = P_{R(W)}P_{R(S)}$. Thus, replacing in (14) we have $\tilde{f} = P_{R(W)}P_{R(S)}f$ which coincide with the solution obtained in Theorem 2, [12]. For equivalent conditions for $(S^*W)^\dagger = W^\dagger(S^*)^\dagger$ see [5].

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