Geometric Interpolation in p-Schatten Class

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Dedicated to the memory of my brother Esteban (1988-2003)

Abstract. The aim of this work is to apply the complex interpolation method to norms of n-tuples of operators in the p-Schatten classes of a Hilbert space H. The norms considered define Finsler metrics in a certain manifold of positive operators, and can be regarded as weighted p-norms, the weight being a positive invertible operator. As a by-product, we obtain Clarkson's type inequalities.

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1. Introduction

Let $B(H)$ denote the algebra of bounded operators acting on a complex and separable Hilbert space H, $Gl(H)$ the group of invertible elements of $B(H)$ and $Gl(H)^+$ the set of all positive elements of $Gl(H)$.
If $X \in B(H)$ is compact we denote by $\{s_j(X)\}$ the sequence of singular values of X (decreasingly ordered). For $1 \leq p < \infty$, let

$$\|X\|_p = (\sum s_j(X)^p)^{1/p} = (tr\,|X|^p)^{1/p},$$

where tr is the usual trace functional, this defines a norm on the set

$$B_p(H) = \{X \in B(H) : \|X\|_p < \infty\},$$

called the $p - Schatten\ class$ of $B(H)$ (to simplify notation we use B_p). By convention $\|X\| = \|X\|_\infty = s_1(X)$. A reference for this subject is [9].

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The Clarkson inequalities on B_p assert that for $1 \le p \le 2$

$$2^{p-1}(\|A\|_p^p + \|B\|_p^p) \le \|A - B\|_p^p + \|A + B\|_p^p \le 2(\|A\|_p^p + \|B\|_p^p), \tag{1.1}$$

and for $2 \le p < \infty$

$$2(\|A\|_p^p + \|B\|_p^p) \le \|A - B\|_p^p + \|A + B\|_p^p \le 2^{p-1}(\|A\|_p^p + \|B\|_p^p), \tag{1.2}$$

The proofs of these inequalities can be found in [17]. These inequalities have useful applications, in particular they imply the uniform convexity of B_p, for $1 < p < \infty$. In [13], the Clarkson' inequalities are generalized to larger classes of functions including the power functions. Bhatia and Kittaneh [4] proved similar inequalities for trace norms on linear combinations of n operators with roots of unity as coefficients.

From now on, for sake of simplicity, we denote with capital letters the elements of B_p and with lower case letters the elements of $Gl(H)^+$.

On B_p we define the following norm associated with $a \in Gl(H)^+$:

$$\|X\|_{p,a} := \left\| a^{-1/2} X a^{-1/2} \right\|_p .$$

The use of this norm has a geometrical meaning which shall be explained later. For instance, we obtain (Theorem 4.1) that:

For $a, b \in Gl(H)^+, X_0, ..., X_{n-1} \in B_p, 1 \le p < \infty$ and $t \in [0,1]$, we have

$$\tilde{k} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \le \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j \|_{p,\gamma_{a,b}(t)}^p \le \tilde{K} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p$$

where $\theta_0, ..., \theta_{n-1}$ are the n roots of unity, for certain constants $\tilde{k} = \tilde{k}(p, a, b, t)$ and $\tilde{K} = \tilde{K}(p, a, b, t)$ which can be computed explicitly.

The material is organized as follows. Section 2 contains a brief summary of the Complex interpolation method. In Section 3 we apply this method and obtain that the curve of interpolation coincides with the curve of weighted norms determined by the positive invertible elements

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2} b a^{-1/2})^t a^{1/2}.$$

This curve naturally appears when studying the geometric structure of the set of positive invertible elements of a C*-algebra [7].

In Section 4, we present an elementary interpolation argument to obtain Clarkson's type inequalities.

Finally, in Section 5 we present the geometrical meaning of the interpolating curve $\gamma_{a,b}$.

2. The Complex Interpolation Method

We recall the construction of interpolation spaces, usually called the complex interpolation method. We follow the notation used in [2] and we refer to [15] and [5] for details on the complex interpolation method.

A compatible couple of Banach spaces is a pair $\bar{X} = (X_0, X_1)$ of Banach spaces X_0, X_1 both continuously embedded in some Hausdorff topological vector space $\mathcal{U}$. Observe that for all $a, b \in Gl(H)^+$ and $1 \leq p < \infty$, the Banach spaces $(B_p, \|\,.\,\|_{p,a})$ and $(B_p, \|\,.\,\|_{p,b})$ are compatible. We will simply write this pair of spaces $\overline{B_p}$ when no confusion can arise.

If X_0 and X_1 are compatible, then one can form their sum $X_0 + X_1$ and their intersection $X_0 \cap X_1$. The sum consists of all $x \in \mathcal{U}$ such that one can write $x = y + z$ for some $y \in X_0$ and $z \in X_1$.

Suppose that X_0 and X_1 are compatible Banach spaces. Then $X_0 \cap X_1$ is a Banach space with their norm defined by

$$\|x\|_{X_0 \cap X_1} = max(\|x\|_{X_0}, \|x\|_{X_1}).$$

Moreover, $X_0 + X_1$ is also a Banach space with the norm

$$\|x\|_{X_0 + X_1} = inf\{\|y\|_{X_0} + \|z\|_{X_1} : x = y + z\}.$$

A Banach space X is said to be an intermediate space with respect to $\bar{X}$ if

$$X_0 \cap X_1 \subset X \subset X_0 + X_1,$$

and both inclusions are continuous.

Given a compatible pair $\bar{X} = (X_0, X_1)$, one considers the space $\mathcal{F}(\bar{X}) = \mathcal{F}(X_0, X_1)$ of all functions f defined in the strip

$$S = \{z \in \mathbb{C} : 0 \leq Re(z) \leq 1\},$$

with values in $X_0 + X_1$, and having the following properties:

1. $f(z)$ is continuous and bounded in norm of $X_0 + X_1$ on the strip S.
2. $f(z)$ is analytic relative to the norm of $X_0 + X_1$ on $\overset{\circ}{S}$.
3. $f(j + iy)$ assumes values in the space X_j $(j = 0, 1)$ and is continuous and bounded in the norm of this space.

One equips the vector space $\mathcal{F}(\bar{X})$ with the norm

$$\|f\|_{\mathcal{F}(\bar{X})} = max\,\{\sup_{y \in \mathbb{R}} \|f(iy)\|_{X_0}, \sup_{y \in \mathbb{R}} \|f(1 + iy)\|_{X_1}\}.$$

The space $(\mathcal{F}(\bar{X}), \|\;\|_{\mathcal{F}(\bar{X})})$ is a Banach space.

For each $0 < t < 1$ the complex interpolation space, associated to the couple $\bar{X}$, $\bar{X}_{[t]} = (X_0, X_1)_{[t]}$ is the set of all elements $x \in X_0 + X_1$ representable in the form

$x = f(t)$ for some functions $f \in \mathcal{F}(\bar{X})$, equipped with the complex interpolation norm

$$\|x\|_{[t]} = inf\{\|f\|_{\mathcal{F}(\bar{X})}\, ;\ f \in \mathcal{F}(\bar{X}),\ f(t) = x\}.$$

The two main results of the theory are:

Theorem A. The space $\bar{X}_{[t]}$ is a Banach space and an intermediate space with respect to $\bar{X}$.

Theorem B. Let $\bar{X}$ and $\bar{Y}$ two compatible couples. Assume that T is a linear operator from X_j to Y_j bounded by M_j, $j = 0, 1$. Then for $t \in [0, 1]$

$$\|T\|_{\bar{X}_{[t]} \to \bar{Y}_{[t]}} \leq M_0^{1-t} M_1^t.$$

3. Geometric Interpolation

In this section, we state the main result of this paper. First, we introduce the notation.
For $1 \leq p < \infty, n \in \mathbb{N}, s \geq 1$ and $a \in Gl(H)^+$, let

$$B_p^{(n)} = \{(X_0, \dots, X_{n-1}) : X_i \in B_p\},$$

endowed with the norm

$$\|(X_0, \dots, X_{n-1})\|_{p,a;s} = (\|X_0\|_{p,a}^s + \dots + \|X_{n-1}\|_{p,a}^s)^{1/s},$$

and $\mathbb{C}^n$ endowed with the norm

$$|(a_0, \dots, a_{n-1})|_s = (|a_0|^s + \dots + |a_{n-1}|^s)^{1/s}.$$

We consider the action of $Gl(H)$ on $B_p^{(n)}$, defined by

$$l : Gl(H) \times B_p^{(n)} \longrightarrow B_p^{(n)},\ l_g((X_0, \dots, X_{n-1})) = (gX_0g^*, \dots, gX_{n-1}g^*). \quad (3.1)$$

From now on, we denote with $B_{p,a;s}^{(n)}$ the space $B_p^{(n)}$ endowed with the norm $\|(., \dots, .)\|_{p,a;s}$.

Proposition 3.1. The norm in $B_{p,a;s}^{(n)}$ is invariant for the action of the group of invertible elements. By this we mean that for each $(X_0, \dots, X_{n-1}) \in B_p^{(n)}$, $a \in Gl(H)^+$ and $g \in Gl(H)$, we have

$$\|(X_0, \dots, X_{n-1})\|_{p,a;s} = \|l_g((X_0, \dots, X_{n-1}))\|_{p,gag^*;s}.$$

Proof. We only prove the case $n = 1$, the other cases are similar.
Let $a \in Gl(H)^+, g \in Gl(H)$ and $X \in B_p$, observe that

$$gXg^* = ga^{1/2}a^{-1/2}Xa^{-1/2}a^{1/2}g^*.$$

Denote by $z = ga^{\frac{1}{2}}$ then

$$(gag^*)^{-\frac{1}{2}} = (ga^{\frac{1}{2}}a^{\frac{1}{2}}g^*)^{-\frac{1}{2}} = (zz^*)^{-\frac{1}{2}} = |z^*|^{-1},$$

therefore
$$(gag^*)^{-\frac{1}{2}} gXg^*(gag^*)^{-\frac{1}{2}} = |z^*|^{-1} za^{-\frac{1}{2}}Xa^{-\frac{1}{2}}z^* |z^*|^{-1}.$$
From the polar decomposition applied to $z \in Gl(H)$, $z = |z^*| \rho_z$ with ρ_z unitary, we have
$$(gag^*)^{-\frac{1}{2}} gXg^*(gag^*)^{-\frac{1}{2}} = \rho_z a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\rho_z^*.$$
Now, since $|srs^*| = s|r|s^*$ for all unitary s, we get
$$\begin{aligned} \|gXg^*\|_{p,gag^*}^p &= tr\left|\rho_z a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\rho_z^*\right|^p = tr(\rho_z \left|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\right|^p \rho_z^*) \\ &= tr\left|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\right|^p = \left\|a^{-\frac{1}{2}}Xa^{-\frac{1}{2}}\right\|_p^p = \|X\|_{p,a}^p. \end{aligned}$$
□

Theorem 3.1. *Let $a, b \in Gl(H)^+, 1 \le p, s < \infty$, $n \in \mathbb{N}$ and $t \in (0,1)$. Then*
$$(B_{p,a;s}^{(n)}, B_{p,b;s}^{(n)})_{[t]} = B_{p,\gamma_{a,b}(t);s}^{(n)}.$$

Proof. Recall Hadamard's classical three line theorem ([19], page 33):
Let $f(z)$ be a Banach space-valued function, bounded and continuous on the strip S, analytic in the interior, satisfying
$$\|f(z)\|_X \le M_0 \ \ if \ \ Re(z) = 0$$
and
$$\|f(z)\|_X \le M_1 \ \ if \ \ Re(z) = 1,$$
where $\|.\|_X$ denotes the norm of the Banach space X. Then
$$\|f(z)\|_X \le M_0^{1-Re(z)} M_1^{Re(z)}.$$
for all $z \in S$.
In order to simplify, we will only consider the case $n = 2$. The proof below works for n-tuples ($n \ge 3$) with obvious modifications.
By the previous proposition, we have that $\|(X_1, X_2)\|_{[t]}$ is equal to the norm of $a^{-1/2}(X_1, X_2)a^{-1/2}$ interpolated between the norms $\|(.,.)\|_{p,1;s}$ and $\|(.,.)\|_{p,c;s}$. Consequently it is sufficient to prove our statement for these two norms.
The proof consists in showing that for all $t \in (0,1)$, $\|(X_1, X_2)\|_{[t]}$ and $\|(X_1, X_2)\|_{p,c^t;s}$ coincide in $B_p^{(2)}$.
Let $t \in (0,1)$ and $(X_1, X_2) \in B_p^{(2)}$ such that $\|(X_1, X_2)\|_{p,c^t;s} = 1$, and define
$$f(z) = c^{\frac{z}{2}} c^{-\frac{t}{2}} (X_1, X_2) c^{-\frac{t}{2}} c^{\frac{z}{2}} = (f_1(z), f_2(z))$$
Then for each $z \in S$, $f(z) \in B_p^{(2)}$
$$\|f(iy)\|_{p,1;s} = \left\|c^{\frac{iy}{2}} c^{-\frac{t}{2}} (X_1, X_2) c^{-\frac{t}{2}} c^{\frac{iy}{2}}\right\|_{p,1;s} = (\sum_{i=1}^{2} \left\|c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_i c^{-\frac{t}{2}} c^{\frac{iy}{2}}\right\|_p^s)^{1/s} \le 1,$$

and
$$\|f(1+iy)\|_{p,c;s} = (\sum_{i=1}^{2} \left\| c^{\frac{1}{2}} c^{\frac{iy}{2}} c^{-\frac{t}{2}} X_i c^{-\frac{t}{2}} c^{\frac{iy}{2}} c^{\frac{1}{2}} \right\|_{p,c}^{s})^{1/s} \leq 1.$$
Since $f(t) = (X_1, X_2)$ and $f = (f_1, f_2) \in \mathcal{F}(B_p^{(2)})$ we have $\|(X_1, X_2)\|_{[t]} \leq 1$. Thus we have shown that
$$\|(X_1, X_2)\|_{[t]} \leq \|(X_1, X_2)\|_{p,c^t;s}.$$
To prove the converse inequality, let $f = (f_1, f_2) \in \mathcal{F}(B_p^{(2)})$; $f(t) = (X_1, X_2)$ and $(Y_1, Y_2) \in B_q^{(2)}$ with $\|Y_k\|_q \leq 1$, where q is the conjugate exponent for $1 < p < \infty$ (or a compact operator and $q = \infty$ if $p = 1$). For $k = 1, 2$, let
$$g_k(z) = c^{-\frac{z}{2}} Y_k c^{-\frac{z}{2}}.$$
Consider the function $h : S \to (\mathbb{C}^2, |(.,.)|_s)$,
$$h(z) = (tr(f_1(z)g_1(z)), tr(f_2(z)g_2(z))).$$
Since $f(z)$ is analytic in $\overset{\circ}{S}$ and bounded in S as a $B_p^{(2)}$- valued function, then h is analytic in $\overset{\circ}{S}$ and bounded in S, and
$$h(t) = (tr(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1, tr(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2)).$$
By Hadamard's three line theorem, applied to h and the Banach space $(\mathbb{C}^2, |(.,.)|_s)$, we have
$$|h(t)|_s \leq max\{\sup_{y\in\mathbb{R}} |h(iy)|_s, \sup_{y\in\mathbb{R}} |h(1+iy)|_s\}.$$
For $j = 0, 1$,
$$\begin{aligned} \sup_{y\in\mathbb{R}} |h(j+iy)|_s &= \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} |tr(f_k(j+iy)g_k(j+iy))|^s)^{1/s} \\ &= \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} \left| tr(c^{-j/2} f_k(j+iy) c^{-j/2} g_k(iy)) \right|^s)^{1/s} \\ &\leq \sup_{y\in\mathbb{R}} (\sum_{k=1}^{2} \|f_k(j+iy)\|_{p,c^j}^s)^{1/s} \leq \|f\|_{\mathcal{F}(B_p^{(2)})}, \end{aligned}$$
then
$$\begin{aligned} \|X_1\|_{p,c^t}^s + \|X_2\|_{p,c^t}^s &= \sup_{\substack{\|Y_1\|_q \leq 1 \\ \|Y_2\|_q \leq 1}} \{ \left| tr(c^{-\frac{t}{2}} X_1 c^{-\frac{t}{2}} Y_1) \right|^s + \left| tr(c^{-\frac{t}{2}} X_2 c^{-\frac{t}{2}} Y_2) \right|^s \} \\ &\leq \sup_{\substack{\|Y_1\|_q \leq 1 \\ \|Y_2\|_q \leq 1}} |h(t)|_s^s \leq \|f\|_{\mathcal{F}(B_p^{(2)})}^s. \end{aligned}$$

Since the previous inequality is valid for each $f \in \mathcal{F}(B_p^{(2)})$ with $f(t) = (X_1, X_2)$, we have

$$\|(X_1, X_2)\|_{p,c^t;s} \leq \|(X_1, X_2)\|_{[t]}\,.$$

□

In the special case $n = s = 1$ we obtain

Corollary 3.2. *Given $a, b \in Gl(H)^+$ and $1 \leq p < \infty$. Then*

$$(B_{p,a}, B_{p,b})_{[t]} = B_{p,\gamma_{a,b}(t)}.$$

Remark 3.1. Note that when a and b commute the curve is given by $\gamma_{a,b}(t) = a^{1-t}b^t$. The previous corollary tells us that the interpolating space, $B_{p,\gamma_{a,b}(t)}$ can be regarded as a weighted p-Schatten space with weight $a^{1-t}b^t$ (see [2], Th. 5.5.3).

The complex interpolation method has been used for different authors in the context of operator algebras. For instance:

1. In 1977, Uhlmann [21] discussed the quadratic interpolation and introduced the relative entropy for states of an operator algebra. His quadratic interpolation is reduced to a path generated by the geometric mean and the relative entropy is the derivative of this path. Corach et al. [8] pointed out that this path can be regarded as a geodesic in a manifold of positive invertible elements with a Finsler norm.
2. The theory of L^p spaces associated with general (not necessarily semifinite) von Neumann algebras has been developed by U. Haagerup [10]. Kosaki [14] obtained these spaces via complex interpolation in a special case, when there exists a normal faithful positive functional ϕ on the von Neumann algebra M.
3. Andruchow et al. proved in [1] that if $A \subset B(H)$ is a C^* algebra, a, b two invertible positive elements in A, and $\| \, \|_a$ and $\| \, \|_b$ the corresponding quadratic norms on H induced by them, i.e. $\|x\|_a = \langle ax, x\rangle$, then the complex interpolation method, is also determined by $\gamma_{a,b}$. This curve is the unique geodesic of the manifold of positive invertible elements of A, which joins a and b.

4. An interpolation technique to obtain Clarkson's type inequalities

Consider the linear operator $T_n : B^{(n)}_{p,a;s} \longrightarrow B^{(n)}_{p,b;s}$ given by

$$T_n(X_0, ..., X_{n-1}) = (\sum_{j=0}^{n-1} X_j, \sum_{j=0}^{n-1} \theta_j^1 X_j, ..., \sum_{j=0}^{n-1} \theta_j^{n-1} X_j),$$

where $\theta_0, ..., \theta_{n-1}$ are the n roots of unity, i.e. $\theta_j = e^{\frac{2\pi i j}{n}}$.

We remark that the inequalities (1.1) and (1.2) can be viewed as statements about the norm of T_2. This approach was used by Klaus ([20], page 22).
We use the same idea and the interpolation method to obtain the following inequalities.

Theorem 4.1. *For* $a, b \in Gl(H)^+, X_0, ..., X_{n-1} \in B_p, 1 \leq p < \infty$ *and* $t \in [0, 1]$, *we have*

$$\tilde{k} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \leq \sum_{k=0}^{n-1} \| \sum_{j=0}^{n-1} \theta_j^k X_j \|_{p,\gamma_{a,b}(t)}^p \leq \tilde{K} \sum_{j=0}^{n-1} \|X_j\|_{p,a}^p \tag{4.1}$$

where

$$\tilde{k} = \tilde{k}(p, a, b, t) \begin{cases} n^{p-1} \left\| b^{1/2} a^{-1} b^{1/2} \right\|^{-pt} & \text{if } 1 \leq p \leq 2, \\ n \left\| b^{1/2} a^{-1} b^{1/2} \right\|^{-pt} & \text{if } 2 \leq p < \infty, \end{cases}$$

and

$$\tilde{K} = \tilde{K}(p, a, b, t) \begin{cases} n \left\| a^{1/2} b^{-1} a^{1/2} \right\|^{pt} & \text{if } 1 \leq p \leq 2, \\ n^{p-1} \left\| a^{1/2} b^{-1} a^{1/2} \right\|^{pt} & \text{if } 2 \leq p < \infty. \end{cases}$$

Proof. We only prove the case $n = 2$ and $1 \leq p \leq 2$, the other cases are similar.
We will denote by $\gamma(t) = \gamma_{a,b}(t)$, when no confusion can arise.
Consider the space $B_p^{(2)}$ with the norm:

$$\|(X, Y)\|_{p,a;p} = (\|X\|_{p,a}^p + \|Y\|_{p,a}^p)^{1/p},$$

where $a \in Gl(H)^+$.
By (1.1) (or [[4], Th. 2] with $n = 2$) the norm of T_2 is at most $2^{1/p}$ when

$$T : (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \longrightarrow (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}),$$

and the norm of T_2 is at most $2^{1/p} \left\| a^{1/2} b^{-1} a^{1/2} \right\|$ when

$$T : (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \longrightarrow (B_p^{(2)}, \|(\,.,.\,)\|_{p,b;p}).$$

Therefore, using the complex interpolation, we obtain the following diagram of interpolation for $t \in [0, 1]$

$$\begin{array}{ccc} & & (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) \\ & \overset{T}{\nearrow} & \\ (B_p^{(2)}, \|(\,.,.\,)\|_{p,a;p}) & \xrightarrow{T_t} & (B_p^{(2)}, \|(\,.,.\,)\|_{p,\gamma(t);p}) \\ & \underset{T}{\searrow} & \\ & & (B_p^{(2)}, \|(\,.,.\,)\|_{p,b;p}). \end{array}$$

By *Theorem B*, T_t satisfies

$$\begin{aligned}\|T_t(X,Y)\|_{p,\gamma(t);p} &\leq (2^{1/p}\|a^{1/2}b^{-1}a^{1/2}\|)^t(2^{1/p})^{1-t}\,\|(X,Y)\|_{p,a;p}\\ &= 2^{1/p}\|a^{1/2}b^{-1}a^{1/2}\|^t\,\|(X,Y)\|_{p,a;p}\,. \end{aligned} \tag{4.2}$$

Now applying the Complex method to

$$\begin{array}{ccc} (B_p^{(2)},\|(\,.,.\,)\|_{p,a;p}) & & \\ & \searrow^{T} & \\ (B_p^{(2)},\|(\,.,.\,)\|_{p,\gamma(t);p}) & \xrightarrow{T_t} & (B_p^{(2)},\|(\,.,.\,)\|_{p,a;p}) \\ & \nearrow_{T} & \\ (B_p^{(2)},\|(\,.,.\,)\|_{p,b;p}) & & \end{array}$$

one obtains

$$\begin{aligned}\|T(X,Y)\|_{p,a;p} &\leq (2^{1/p}\|b^{1/2}a^{-1}b^{1/2}\|)^t(2^{1/p})^{1-t}\,\|(X,Y)\|_{p,\gamma(t);2}\\ &= 2^{1/p}\|b^{1/2}a^{-1}b^{1/2}\|^t\,\|(X,Y)\|_{p,\gamma(t);2}\,. \end{aligned} \tag{4.3}$$

Replacing in (4.3) $X=\frac{Z+W}{2}$ and $Y=\frac{Z-W}{2}$ we obtain

$$\|Z\|_{p,a}^p+\|W\|_{p,a}^p\leq 2^{1-p}\|b^{1/2}a^{-1}b^{1/2}\|^{pt}(\|Z-W\|_{p,\gamma(t)}^p+\|Z+W\|_{p,\gamma(t)}^p), \tag{4.4}$$

or equivanletly

$$2^{p-1}\|b^{1/2}a^{-1}b^{1/2}\|^{-pt}(\|Z\|_{p,a}^p+\|W\|_{p,a}^p)\leq\|Z-W\|_{p,\gamma(t)}^p+\|Z+W\|_{p,\gamma(t)}^p\,. \tag{4.5}$$

Finally, the inequalities (4.2) and (4.5) complete the proof. □

Theorem 4.2. *For $a,b\in Gl(H)^+, X_0,...,X_{n-1}\in B_p, 1\leq p<\infty$ and $t\in[0,1]$, we have*

$$k\sum_{j=0}^{n-1}\|X_j\|_{p,a}^2\leq\sum_{k=0}^{n-1}\|\sum_{j=0}^{n-1}\theta_j^kX_j\|_{p,\gamma_{a,b}(t)}^2\leq K\sum_{j=0}^{n-1}\|X_j\|_{p,a}^2\,, \tag{4.6}$$

where

$$k=k(p,a,b,t)\begin{cases} n^{2-2/p}\left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{if } 1\leq p\leq 2,\\ n^{2/p}\left\|b^{1/2}a^{-1}b^{1/2}\right\|^{-2t} & \text{if } 2\leq p<\infty,\end{cases}$$

and

$$K=K(p,a,b,t)\begin{cases} n^{2/p}\left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{if } 1\leq p\leq 2,\\ n^{2-2/p}\left\|a^{1/2}b^{-1}a^{1/2}\right\|^{2t} & \text{if } 2\leq p<\infty.\end{cases}$$

Proof. A slight change in the previous proof proves our statement.
We need to consider the space $B_p^{(n)}$ endowed with the norm

$$\|(X_0, ..., X_{n-1})\|_{p,a;2} = (\|X_0\|_{p,a}^2 + .. + \|X_{n-1}\|_{p,a}^2)^{1/2},$$

where $a \in Gl(H)^+$ and [4], Th. 1. □

5. The geometry of Δ_p^1

In this section we give a geometric context to what has been previously presented. More precisely we prove that the curves $\gamma_{a,b}$ are minimal curves of a Finsler geometry for a manifold of positive and invertible operators.

5.1. Topological and differentiable structure of Δ_p^1.

Consider for $1 \le p < \infty$ the following set of Fredholm operators,

$$\mathcal{L}_p = \{\lambda + X \in B(H) : \lambda \in \mathbb{C},\ X \in B_p\}.$$

$\mathcal{L}_p$ is a complex linear subalgebra consisting of the p-Schatten class perturbations of multiples of the identity. There is a natural norm for this subspace

$$\|\lambda + X\|_{(p)} = |\lambda| + \|X\|_p.$$

Lemma 5.1. Let $\lambda + X, \mu + Y \in \mathcal{L}_p$. Then

1. $\|\lambda + X\| \le \|\lambda + X\|_{(p)}$,
2. $\|(\lambda + X)(\mu + Y)\|_{(p)} \le \|\lambda + X\|_{(p)}\|\mu + Y\|_{(p)}$.

In particular, $(\mathcal{L}_p, +, .)$ is a Banach algebra.

Proof. The proof is straightforward. □

The selfadjoint part of $\mathcal{L}_p$ is

$$\mathcal{L}_p^{sa} = \{\lambda + X \in \mathcal{L}_p : (\lambda + X)^* = \lambda + X\},$$

Remark 5.1.
1. $(\mathcal{L}_p, \|.\|_{(p)})$ is the unitization of $(B_p, \|.\|_p)$.
2. Note that the multiples of the identity $\lambda 1$ and the operators $X \in B_p$ are linearly independent. Therefore

$$\lambda + X \in \mathcal{L}_p^{sa} \text{ if and only if } \lambda \in \mathbb{R}\ ,\ X^* = X.$$

   Formally,

$$\mathcal{L}_p = \mathbb{C} \oplus B_p \qquad\qquad \mathcal{L}_p^{sa} = \mathbb{R} \oplus B_p^{sa},$$

   where B_p^{sa} denotes the set of selfadjoint p-Schatten class operators.

Inside $\mathcal{L}_p^{sa}$, we consider

$$\Delta_p = \{\lambda + X \in \mathcal{L}_p : \ \lambda + X > 0\} \subset Gl(H)^+.$$

and

$$\Delta_p^1 = \{1 + X \in \mathcal{L}_p : \ 1 + X > 0\}.$$

Apparently Δ_p is an open subset of $\mathcal{L}_p^{sa}$, and therefore a differentiable (analytic) submanifold.

The next step is to prove that Δ_p^1 is a submanifold of Δ_p. For this purpose, we consider

$$\theta : \Delta_p \to \mathbb{R} \ , \ \theta(\lambda + X) = \lambda.$$

Lemma 5.2. θ is a submersion.

Proof. It is sufficient to show that $d\theta_{\lambda+a}$ is surjective and $ker(d\theta_{\lambda+a})$ is complemented ([16], Th. 2.2).

Since $\mathcal{L}_p^{sa}$ and $\mathbb{R}$ are Banach spaces and θ is a continuous linear map we get that $d\theta_{\lambda+X} = \theta$

Apparently, $d\theta_{\lambda+X}$ is suryective and $ker(d\theta_{\lambda+X})$ has codimension 1 and hence is complemented. □

It follows that Δ_p^1 is a submanifold, since $\Delta_p^1 = \theta^{-1}(\{1\})$. These facts imply that, for $1 + X \in \Delta_p^1$, $(T\Delta_p^1)_{1+X}$ identifies with B_p^{sa}.

Remark 5.1. In a previous work we studied the geometry of Δ_1^1, see [6].

Let $Gl(H, B_p)$ be the subgroup of $Gl(H)$, consisting of invertible p-Schatten class perturbations of the identity, i.e.

$$Gl(H, B_p) = \{1 + X \in Gl(H) : X \in B_p\} = \{g \in Gl(H) : g - 1 \in B_p\} \subseteq Gl(H).$$

The group $Gl(H, B_p)$ is a Banach-Lie group. A classical reference for this subject is [11].

There is a natural action of $Gl(H, B_p)$ on Δ_p^1, defined as the restriction of l on $Gl(H, B_p) \times B_p^{(n)}$ as in (3.1).

This action is clearly differentiable and transitive, since if $1 + X, 1 + Y \in \Delta_p^1$ then

$$l_r(1 + X) = (1 + Y),$$

for $r = (1 + Y)^{\frac{1}{2}}(1 + X)^{-\frac{1}{2}} \in Gl(H, B_p)$.

If $1 + Y \in \Delta_p^1$, we define the length of a tangent vector $X \in (T\Delta_p^1)_{1+Y}$ by

$$\|X\|_{p,1+Y} = \left\|(1 + Y)^{-\frac{1}{2}} X (1 + Y)^{-\frac{1}{2}}\right\|_p .$$

By Proposition 3.1 the Finsler norm is invariant for the action of $Gl(H, B_p)$.

5.2. Minimal Curves

In this section we investigate the existence of minimal curves for the Finsler metric just defined. The expresion “minimal” is understood in terms of the length functional (or more generally q-energy functional). We prove that the interpolating curve $\gamma_{a,b}$ joining a with b is the minimun of the q-energy functional for $q \geq 1$. We observe that this curve looks formally equal to the geodesic between positive definitive matrices (regarded as a symmetric space, see [18]).
For a piecewise differentiable curve $\alpha : [0,1] \to \Delta_p^1$, one computes the *length* of the curve α by

$$length(\alpha) = \int_0^1 \|\dot{\alpha}(t)\|_{p,\alpha(t)} dt.$$

Proposition 5.1. Given a, b in Δ_p^1, the curve $\gamma_{a,b} : [0,1] \longrightarrow \Delta_p^1$ has length $\|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_p$.

Proof. Since the group $Gl(H, B_p)$ acts isometrically and transitively on Δ_p^1, it suffices to prove the theorem for $a = 1$. Then

$$\|\dot{\gamma_{1,b}}(t)\|_{p,\gamma_{1,b}(t)} = \|log(b)b^t\|_{p,b^t} = \|b^{t/2} log(b) b^{-t/2}\|_p = \|log(b)\|_p,$$

because $log(b)$ and b^t conmute for every $t \in \mathbb{R}$. □

Definition 5.2. Let $a, b \in \Delta_p^1$. We denote

$$\Omega_{a,b} = \{\alpha : [0,1] \to \Delta_p^1 : \alpha \text{ is a } \mathcal{C}^1 \text{ curve, } \alpha(0) = a \text{ and } \alpha(1) = b\}.$$

As in classical differential geometry, we consider the geodesic distance between a and b (in the Finsler metric) defined by

$$d(a,b) = inf\{length(\alpha) : \alpha \in \Omega_{a,b}\}.$$

The next step consists in showing that $\gamma_{a,b}$ are short curves, i.e. if $\delta \in \Omega_{a,b}$ then

$$length(\gamma_{a,b}) \leq length(\delta).$$

and hence

$$d(a,b) = \|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_p.$$

The proof of this fact requires some preliminaries. We begin with the following inequalities (see [12]):
Let A, B, X be Hilbert space operators with $A, B \geq 0$. For any unitarily invariant norm $|||.|||$ we have

$$|||A^{1/2} X B^{1/2}||| \leq ||| \int_0^1 A^t X B^{1-t} dt||| \leq \frac{1}{2}|||AX + XB|||. \tag{5.1}$$

The proof of the next inequality, called by Bhatia (in the context of matrices) the *exponential metric increasing property*, is based on a similar argument used in [3].

Proposition 5.2. For all $X, Y \in B_p^{sa}$

$$\|Y\|_p \leq \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_p,$$

where $dexp_X$ denotes the differential of the exponential map at X.

Proof. The proof is based on the inequality (5.1) and the formula below:

Claim 5.1. $dexp_X(Y) = \int_0^1 e^{tX} Y e^{(1-t)X} dt.$

We provide here a simple proof of this equality. Since

$$\frac{d}{dt}(e^{tX} e^{(1-t)Y}) = e^{tX}(X - Y)e^{(1-t)Y},$$

we have

$$e^X - e^Y = \int_0^1 e^{tX}(X - Y)e^{(1-t)Y} dt,$$

and hence

$$\lim_{h \to 0} \frac{e^{X+hY} - e^X}{h} = \int_0^1 e^{tX} Y e^{(1-t)X} dt.$$

Let $X, Y \in B_p^{sa}$. Write $Y = e^{\frac{X}{2}}(e^{-\frac{X}{2}} Y e^{-\frac{X}{2}})e^{\frac{X}{2}}$. Then using the inequalities (5.1) we obtain

$$\begin{aligned} \|Y\|_p &\leq \|\int_0^1 e^{tX}(e^{-\frac{X}{2}} Y e^{-\frac{X}{2}})e^{(1-t)X} dt\|_p = \|e^{-\frac{X}{2}} \int_0^1 e^{tX} Y e^{(1-t)X} dt e^{-\frac{X}{2}}\|_p \\ &= \|e^{-\frac{X}{2}} dexp_X(Y) e^{-\frac{X}{2}}\|_p. \end{aligned}$$

This proves the proposition. □

We are now ready to prove the main result in this section.

Theorem 5.3. *Let $a, b \in \Delta_p^1$, then $\gamma_{a,b}$ is the shortest curve joining them. So*

$$d(a, b) = \|log(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})\|_p.$$

Proof. Since the group $Gl(H, B_p)$ acts isometrically and transitively on Δ_p^1, it is sufficient to prove the statement for $a = 1$. Then

$$\gamma_{1,b} = b^t = e^{t\, logb} \qquad \text{and} \qquad length(\gamma_{1,b}) = \|log\ b\|_1.$$

Let $\gamma \in \Omega_{1,b}$; so write $\gamma(t) = e^{\alpha(t)}$ we get

$$\begin{aligned} \|\gamma(t)^{-\frac{1}{2}} \dot{\gamma}(t) \gamma(t)^{-\frac{1}{2}}\|_p &= \|e^{-\frac{\alpha(t)}{2}} e^{\dot{\alpha}(t)} e^{-\frac{\alpha(t)}{2}}\|_p = \|e^{-\frac{\alpha(t)}{2}} dexp_{\alpha(t)}(\dot{\alpha}(t)) e^{-\frac{\alpha(t)}{2}}\|_p \\ &\geq \|\dot{\alpha}(t)\|_p. \end{aligned}$$

Finally,

$$\begin{aligned} length(\gamma) &= \int_0^1 \|\dot{\gamma}(t)\|_{p,\gamma(t)} dt = \int_0^1 \|\gamma(t)^{-\frac{1}{2}} \dot{\gamma}(t) \gamma(t)^{-\frac{1}{2}}\|_p dt \geq \int_0^1 \|\dot{\alpha}(t)\|_p dt \\ &\geq \|\int_0^1 \dot{\alpha}(t) dt\|_p = \|\alpha(t)|_0^1\|_p = \|\alpha(1) - \alpha(0)\|_p = \|log\ b\|_p. \end{aligned}$$

□

Definition 5.4. For every $q \in \mathbb{R} - \{0\}$ we define the *q-energy functional*

$$E_q : \Omega_{a,b} \to \mathbb{R}^+, \qquad E_q(\alpha) := \int_0^1 \|\dot{\alpha}(t)\|^q_{p,\alpha(t)} dt.$$

Remark 5.2. 1. For $q = 1$ we obtain the *length functional*

$$l(\alpha) := \int_0^1 \|\dot{\alpha}(t)\|_{p,\alpha(t)} dt,$$

and for $q = 2$ we obtain the *energy functional*

$$E(\alpha) := \int_0^1 \|\dot{\alpha}(t)\|^2_{p,\alpha(t)} dt,$$

2. For any curve α such that $\|\dot{\alpha}(t)\|_{p,\alpha(t)}$ is constant we have

$$E_q(\alpha) = (\textit{length}(\alpha))^q = (E(\alpha))^{\frac{q}{2}}.$$

In Theorem 5.3 we proved that the curve between a and b minimizes the length functional. This fact is valid also for the q-energy functional (associated with $\Omega_{a,b}$) for $q \in (1, \infty)$.

Proposition 5.3. Let $a, b \in \Delta^1_p$ and $q \in [1, \infty)$. Then the q-energy functional

$$E_q : \Omega_{a,b} \to \mathbb{R}^+, \qquad E_q(\alpha) := \int_0^1 \|\dot{\alpha}(t)\|^q_{p,\alpha(t)} dt,$$

achieves its global minimun $d^q(a, b)$ precisely at $\gamma_{a,b}$.

Proof. Now, let $\alpha \in \Omega_{a,b}$ and $q \in (1, \infty)$ then by Hölder's inequality

$$(\textit{length}(\alpha))^q = (\int_0^1 \|\dot{\alpha}(t)\|_{p,\alpha(t)} dt)^q \leq \int_0^1 \|\dot{\alpha}(t)\|^q_{p,\alpha(t)} dt = E_q(\alpha).$$

On the other hand, $(\textit{length}(\gamma_{a,b}))^q = E_q(\gamma_{a,b})$. This implies that

$$E_q(\gamma_{a,b}) = (\textit{length}(\gamma_{a,b}))^q \leq (\textit{length}(\alpha))^q \leq E_q(\alpha).$$

□

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